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ScienceDirect
Journal of Complexity 22 (2006) 894–908
Journal of COMPLEXITY
www.elsevier.com/locate/jco

# Effective continuities on effective topological spaces

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Received 12 December 2005; accepted 9 June 2006
Available online 10 August 2006

**Abstract**

We extend a notion of effective continuity due to Mori, Tsujii and Yasugi to real-valued functions on effective topological spaces. Under reasonable assumptions, Type-2 computability of these functions is characterized as sequential computability and the effective continuity. We investigate effective uniform topological spaces with a separating set, and adapt the above result under some assumptions. It is also proved that effective local uniform continuity implies effective continuity under the same assumptions.


*Keywords:* Effective topological space; Effective uniform topological space; Effective continuity; Effective local uniform continuity

## 1. Introduction

On the real line, computability of real-valued functions defined by Type-2 computability induced by an admissible representation is equivalent to the one defined by sequential computability and effective uniform continuity [3]. A different situation occurs when these computabilities are adapted on the Fine metric space. Three different computabilities are introduced on the space: uniform and local uniform Fine computability by Mori [5], and Fine computability by Brattka [1].

In [1], Brattka introduced Fine computability by using Fine representation, which is admissible w.r.t. the Fine metric. He characterized this computability as sequential computability and effective continuity. A similar notion of computability for functions on effective uniform topological spaces can be found in [9]. In [10], these three computabilities on the Fine metric space are generalized to effective uniform topological spaces. But they have not been compared with Type-2 computability. This paper aims to develop the relations among sequential computability, effective

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doi:10.1016/j.jco.2006.06.004

continuity due to Mori, Tsujii and Yasugi and Type-2 computability of real-valued functions on effective uniform topological spaces, or more generally, on effective topological spaces.

Section 2 is preliminaries. In Section 3, we extend a notion of effective continuity due to Mori, Tsujii and Yasugi on effective topological spaces. Under some assumptions which mainly state the existence of "effective separating set", we characterize Type-2 computability of real-valued functions on effective topological spaces as sequential computability and the effective continuity. In Section 4, effective topological spaces are naturally derived from effective uniform topological spaces with a separating set. To obtain some effectivities of inclusion and membership relations, we propose another assumption. As a typical example, computable metric spaces are discussed and shown to satisfy the assumption. In Section 5, we define computability of sequences of an effective uniform topological space by using effective convergence. It is proved that the set of the computable sequences forms a computability structure, and that Type-2 computable sequences provide the same set of sequences. In Section 6, we define three different notions of effective continuity on effective uniform topological spaces. We assume the computability of finite union of neighborhoods to compare these effective continuities. The effective continuities and their relations discussed in Section 6 are as follows:

Effective uniform continuity
$\Rightarrow$ Effective local uniform continuity
$\Rightarrow$ Effective continuity
$\Leftrightarrow$ Effective continuity defined in Section 3.

## 2. Preliminaries

For a set $X$, the power set of $X$ is denoted by $\mathcal{P}(X)$, and the set of non-empty finite subsets of $X$ is denoted by $\mathcal{F}(X)$. We denote a partial function $f$ from $X$ to $Y$ by $f : X \rightharpoonup Y$.

Let $\Sigma$ be an alphabet which contains 0 and 1. For a word $w$, the length of $w$ is denoted by $|w|$. For an infinite sequence $p \in \Sigma^\omega$, the prefix of $p$ of length $m$ is denoted by $p_{<m}$. For a word $w$ and a word or an infinite sequence $p$, we write $w \lhd p$, if $w$ is a subword of $p$, and $w \sqsubseteq p$, if $w$ is a prefix of $p$.

We use the notions and notations in [11] for the representation-based approach to computable analysis. Let $\iota : \Sigma^* \to \Sigma^*$ be a wrapping function. For $I \in \mathbb{N}$, let $v^I_{\mathbb{N}}$ be a standard notation of $\mathbb{N}^I$. Let $\rho_{\mathrm{E}}$ be a standard admissible representation of $\mathbb{R}$ w.r.t. the Euclidean topology. For a set $X$ and a representation $\rho$ of $X$, a sequence $\{x_i\}_{i\in\mathbb{N}^I} \subseteq X$ is called *$\rho$-computable*, if $\mathbb{N}^I \ni i \to x_i \in X$ is $(v^I_{\mathbb{N}}, \rho)$-computable, and a function $f : X \to \mathbb{R}$ is called *sequentially $\rho$-computable*, if $f$ maps each $\rho$-computable sequence of $X$ to a $\rho_{\mathrm{E}}$-computable sequence.

**Remark 2.1.** A sequence $\{x_i\}_{i\in\mathbb{N}^I} \subseteq \mathbb{R}$ is $\rho_{\mathrm{E}}$-computable if and only if there is a recursive sequence $\{r_{i,k}\}_{i\in\mathbb{N}^I,\, k\in\mathbb{N}} \subseteq \mathbb{Q}$ such that

$$\forall i \in \mathbb{N}^I, \quad \forall k \in \mathbb{N}, \quad |x_i - r_{i,k}| \leqslant 2^{-k}.$$

A $\rho_{\mathrm{E}}$-computable sequence is also called a *computable sequence of reals* [7].

## 3. Effective topological spaces

In this section, we introduce effective continuity on effective topological spaces (Definition 3.4). Under Assumption 3.1, which mainly states the existence of "effective separating set", we characterize Type-2 computability of real-valued functions on effective topological spaces as sequential computability and effective continuity (Theorem 3.1).

Note that in this section, we use sequential computability defined in Section 2 by Type-2 computability. This will be developed in Section 5 on effective uniform topological spaces.

We start with recalling the definition of effective topological spaces.

**Definition 3.1** (*Effective topological space, Weihrauch [11]*). A triple $\mathbf{S} = (X, \sigma, \nu)$ is called an *effective topological space*, if $X$ is a non-empty set, $\sigma \subseteq \mathcal{P}(X)$ is a countable system of subsets of $X$ such that

$$\forall x, y \in X, \quad x = y \quad \Leftrightarrow \quad \{A \in \sigma; x \in A\} = \{A \in \sigma; y \in A\}$$

and $\nu : \Sigma^* \rightharpoonup \sigma$ is a notation of $\sigma$. The topology on $X$ generated by $\sigma$ as a subbase is denoted by $\tau_{\mathbf{S}}$.

In the rest of this section, we assume that $\mathbf{S} = (X, \sigma, \nu)$ is an effective topological space.

**Definition 3.2** (*Representation $\delta'_{\mathbf{S}}$ of $X$, Weihrauch [11]*). Define a representation $\delta'_{\mathbf{S}} : \Sigma^\omega \rightharpoonup X$ by

$$\delta'_{\mathbf{S}}(p) = x :\Leftrightarrow \{w; \iota(w) \lhd p\} = \{w \in \mathrm{dom}(\nu); x \in \nu(w)\}.$$

$\delta'_{\mathbf{S}}$ is the restriction of the standard representation of $\mathbf{S}$ to the set of "complete names". It turns out that $\delta'_{\mathbf{S}}$ has an affinity for effective continuity defined below or effective convergence defined in Section 5. So we use this representation for Type-2 computability.

**Remark 3.1.** A sequence $\{x_i\}_{i \in \mathbb{N}^I} \subseteq X$ is $\delta'_{\mathbf{S}}$-computable if and only if $x_i \in \nu(w)$ is an r.e. relation of $i \in \mathbb{N}^I$ and $w \in \Sigma^*$.

**Definition 3.3** (*Countable base $\beta$ of $(X, \tau_{\mathbf{S}})$*).

(1) Define a countable base $\beta$ of $(X, \tau_{\mathbf{S}})$ by

$$\beta := \left\{ \bigcap_{n=1}^{N} A_n ; N \geqslant 0,\ A_1, \ldots, A_N \in \sigma \right\}.$$

(2) Define a notation $\nu_\beta : \Sigma^* \rightharpoonup \beta$ by

$$\nu_\beta(w) = A :\Leftrightarrow \begin{cases} \exists\, w_1, \ldots, w_N \in \mathrm{dom}(\nu)\ (N \geqslant 0) \text{ such that} \\ w = \iota(w_1) \cdots \iota(w_N) \text{ and } A = \bigcap_{n=1}^{N} \nu(w_n). \end{cases}$$

Note that $\nu_\beta(w) = X$ if $w$ is the empty word.

We propose effective continuity on the effective topological space **S** similarly as in [6, Definition 3.1(ii)]. Since $\beta$ is a base of $(X, \tau_{\mathbf{S}})$, for each continuous function $f : X \to \mathbb{R}$, $k \in \mathbb{N}$ and $x_0 \in X$, there is $A \in \beta$ such that $x_0 \in A$ and $|f(x) - f(y)| \leqslant 2^{-k}$ for all $x, y \in A$. The following definition is an effectivization of this property.

**Definition 3.4** (*Effective continuity w.r.t. $\beta$*). A function $f : X \to \mathbb{R}$ is called *effectively continuous* (*EC*) *w.r.t. $\beta$*, if there is a recursive function $\gamma : \mathbb{N}^2 \to \Sigma^*$ such that

(1) $\forall j, k \in \mathbb{N},\ \gamma(j, k) \in \mathrm{dom}(v_\beta)$,
(2) $\forall k \in \mathbb{N},\ \bigcup_{j\in\mathbb{N}} v_\beta(\gamma(j, k)) = X$,
(3) $\forall j, k \in \mathbb{N},\ \forall x, y \in v_\beta(\gamma(j, k)),\ |f(x) - f(y)| \leqslant 2^{-k}$.

**Remark 3.2.**

(1) If $\mathrm{dom}(v_\beta)$ is r.e. and $B'$ is taken as a canonical numbering of the set of all non-empty open intervals with rational endpoints, Spreen's effective continuity [8, Definition 22] is strictly stronger than ours. Indeed, his effective continuity implies Type-2 computability [8, Proposition 25] but ours does not, because if a function $f : X \to \mathbb{R}$ is EC in our sense, then so is $f + c$ for any $c \in \mathbb{R}$.
(2) For the same reason, Hertling's effective continuity [4, Definition 4] is also strictly stronger than ours when the domain $X$ is a semicomputable and recursively separable metric space (see [4, Proposition 10]).
(3) Note also that our definition of effective continuity can be extended to functions to computable metric spaces, or more generally, to effective uniform topological spaces defined in Section 4.

In this paper, we mainly consider the effective topological spaces satisfying the following assumptions:

**Assumption 3.1.**

(1) $\mathrm{dom}(v)$ is r.e.
(2) There is a $\delta'_{\mathbf{S}}$-computable sequence $\{e_i\}_{i\in\mathbb{N}} \subseteq X$ which is dense in $(X, \tau_{\mathbf{S}})$.

The sequence $\{e_i\}$ in (2) of Assumption 3.1 is called an *effective separating set*. Under Assumption 3.1, we characterize $(\delta'_{\mathbf{S}}, \rho_{\mathrm{E}})$-computability as sequential $\delta'_{\mathbf{S}}$-computability and EC w.r.t. $\beta$. The idea of the proof is similar to the one of [1, Theorem 13].

**Theorem 3.1.** *Under Assumption* 3.1, *for each function $f : X \to \mathbb{R}$, $f$ is $(\delta'_{\mathbf{S}}, \rho_{\mathrm{E}})$-computable if and only if $f$ is sequentially $\delta'_{\mathbf{S}}$-computable and EC w.r.t. $\beta$.*

**Proof.** (If part): Suppose that $f : X \to \mathbb{R}$ is sequentially $\delta'_{\mathbf{S}}$-computable and EC w.r.t. $\beta$. Let $\gamma : \mathbb{N}^2 \to \Sigma^*$ be a recursive function as in Definition 3.4. Since the sequence $\{e_i\}$ is $\delta'_{\mathbf{S}}$-computable, there is a $(v_{\mathbb{N}}, \mathrm{id}_{\Sigma^\omega})$-computable function $p_e : \mathbb{N} \to \Sigma^\omega$ such that, for each $i \in \mathbb{N}$, $p_e(i)$ is a $\delta'_{\mathbf{S}}$-name of $e_i$. Since $f$ is sequentially $\delta'_{\mathbf{S}}$-computable, the sequence $\{f(e_i)\}_{i\in\mathbb{N}}$ is $\rho_{\mathrm{E}}$-computable, and so by Remark 2.1, there is a recursive sequence $\{r_{i,k}\}_{i,k\in\mathbb{N}} \subseteq \mathbb{Q}$ such that

$$\forall i, k \in \mathbb{N}, \quad \left|f(e_i) - r_{i,k}\right| \leqslant 2^{-k}.$$

Define a Type-2 machine $M$ by [1]

$M$ := "For inputs $p \in \Sigma^\omega$ and $k \in \mathbb{N}$;

1. Find $\langle j, n\rangle \in \mathbb{N}$ such that $\iota(w) \lhd p_{<n}$ for each $\iota(w) \lhd \gamma(j, k)$.
2. Find $\langle i, m\rangle \in \mathbb{N}$ such that $\iota(w) \lhd p_e(i)_{<m}$ for each $\iota(w) \lhd \gamma(j, k)$.
3. Output $r_{i,k}$ and halt."

Let $x \in X$, $p$ be a $\delta'_{\mathbf{S}}$-name of $x$ and $k \in \mathbb{N}$. Run $M$ on inputs $p$ and $k$. By the property (2) of Definition 3.4, there is $j \in \mathbb{N}$ such that $x \in \nu_\beta(\gamma(j, k))$. Then $x \in \nu(w)$ for each $\iota(w) \lhd \gamma(j, k)$. So for large enough $n \in \mathbb{N}$, $\iota(w) \lhd p_{<n}$ for each $\iota(w) \lhd \gamma(j, k)$. Hence, $M$ can go to Stage 2. By the density of $\{e_i\}$, there is $i \in \mathbb{N}$ such that $e_i \in \nu_\beta(\gamma(j, k))$. So by the same way, $M$ can go to Stage 3. In Stage 3, $x, e_i \in \nu_\beta(\gamma(j, k))$. So by property (3) of Definition 3.4,

$$\begin{aligned}\left|f(x) - r_{i,k}\right| &\leqslant |f(x) - f(e_i)| + \left|f(e_i) - r_{i,k}\right| \\ &\leqslant 2^{-k} + 2^{-k} = 2^{-k+1}.\end{aligned}$$

Thus, for each input $p \in \Sigma^\omega$ and $k \in \mathbb{N}$, if $p \in \mathrm{dom}(\delta'_{\mathbf{S}})$, then $M$ outputs $\varphi(p, k) \in \mathbb{Q}$ such that $\left|f(\delta'_{\mathbf{S}}(p)) - \varphi(p, k)\right| \leqslant 2^{-k+1}$. This proves that $f$ is $(\delta'_{\mathbf{S}}, \rho_{\mathrm{E}})$-computable.

(Only if part): Suppose that $f : X \to \mathbb{R}$ is $(\delta'_{\mathbf{S}}, \rho_{\mathrm{E}})$-computable. Sequential $\delta'_{\mathbf{S}}$-computability is trivial. So we show that $f$ is EC w.r.t. $\beta$. There is a Type-2 machine $M$ as the following: for each input $p \in \Sigma^\omega$ and $k \in \mathbb{N}$, if $p \in \mathrm{dom}(\delta'_{\mathbf{S}})$, then $M$ outputs $\varphi(p, k) \in \mathbb{Q}$ such that $\left|f(\delta'_{\mathbf{S}}(p)) - \varphi(p, k)\right| \leqslant 2^{-k-1}$. Define a Turing machine $N$ by

$N$ := "For inputs $w \in \Sigma^*$ and $k \in \mathbb{N}$;

1. If $w \in \mathrm{dom}(\nu_\beta)$,
2. Simulate $M$ on inputs $w0^\omega$ and $k$ for $|w|$ steps.
3. If $M$ outputs some value and halt,
4. Output 0 and halt.
5. Otherwise, do not halt."

By (1) of Assumption 3.1, $\mathrm{dom}(\nu_\beta)$ is r.e. So if the input $w$ is in $\mathrm{dom}(\nu_\beta)$, $N$ can go to Stage 2. Let $\psi : \Sigma^* \times \mathbb{N} \rightharpoonup \mathbb{N}$ be the partial recursive function computed by $N$. Then there is a recursive function $\gamma : \mathbb{N}^2 \to \Sigma^*$ such that

$$\forall k \in \mathbb{N}, \quad \{\gamma(j, k); j \in \mathbb{N}\} = \left\{w \in \Sigma^*; (w, k) \in \mathrm{dom}(\psi)\right\}.$$

We show that $\gamma$ satisfies properties (1)–(3) of Definition 3.4.

(1) If $w \notin \mathrm{dom}(\nu_\beta)$, $N$ does not halt on the inputs $w$ and $k$. So $(w, k) \in \mathrm{dom}(\psi)$ implies $w \in \mathrm{dom}(\nu_\beta)$. Therefore $\gamma(j, k) \in \mathrm{dom}(\nu_\beta)$ for each $j, k \in \mathbb{N}$.

(2) Let $x \in X$, $p$ be a $\delta'_{\mathbf{S}}$-name of $x$ and $k \in \mathbb{N}$. Then for large enough $m \in \mathbb{N}$, $M$ can output $\varphi(p, k)$ and halt in $m$ steps on inputs $p$ and $k$. Let $w := p_{<m}$. We can assume that $w \in \mathrm{dom}(\nu_\beta)$. Then $x \in \nu_\beta(w)$ and $(w, k) \in \mathrm{dom}(\psi)$. So by the definition of $\gamma$, there is $j \in \mathbb{N}$ such that $\gamma(j, k) = w$. Then $x \in \nu_\beta(\gamma(j, k))$. Therefore, $\bigcup_{j \in \mathbb{N}} \nu_\beta(\gamma(j, k)) = X$ for each $k \in \mathbb{N}$.

(3) Let $j, k \in \mathbb{N}$ and $w := \gamma(j, k)$. Then by the definition of $\gamma$, $(w, k) \in \mathrm{dom}(\psi)$. Let $x, y \in \nu_\beta(w)$. Then there are $\delta'_{\mathbf{S}}$-names $p$ and $q$ of $x$ and $y$, respectively, which satisfy $w \sqsubseteq p, q$.

[1] For convenience, we use integers and rationals as inputs or outputs of Type-2 machines instead of using their notations.

By the definition of $\psi$, $M$ outputs the same value for inputs $p$, $k$ and for inputs $q$, $k$, i.e., $\varphi(p,k) = \varphi(q,k)$. Hence

$$\begin{aligned}|f(x) - f(y)| &= \left|f(\delta'_{\mathbf{S}}(p)) - f(\delta'_{\mathbf{S}}(q))\right| \\ &\leqslant \left|f(\delta'_{\mathbf{S}}(p)) - \varphi(p,k)\right| + \left|\varphi(q,k) - f(\delta'_{\mathbf{S}}(q))\right| \\ &\leqslant 2^{-k-1} + 2^{-k-1} = 2^{-k}.\end{aligned}$$

Therefore, $|f(x) - f(y)| \leqslant 2^{-k}$ for each $j, k \in \mathbb{N}$ and $x, y \in \nu_\beta(\gamma(j,k))$. $\square$

## 4. Effective uniform spaces

In the rest of this paper, we investigate effective uniform topological spaces with a separating set. In this section, natural effective topological spaces are defined which derived from effective uniform topological spaces (Definition 4.2). To obtain some effectivities of inclusion and membership relations, we propose an assumption (Assumption 4.2). As a typical example, computable metric spaces are discussed (Example 4.2). Under the assumption, we prove the existence of effective separating set (Proposition 4.1).

First, we recall the definition of effective uniformities and effective uniform topological spaces [9,12].

**Definition 4.1** (*Effective uniformity, Yasugi et al. [12]*). Let $X$ be a non-empty set and $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of maps $V_n : X \to \mathcal{P}(X)$. $\{V_n\}$ is called an *effective uniformity*, if there are recursive functions $\alpha_1 : \mathbb{N}^2 \to \mathbb{N}$ and $\alpha_2, \alpha_3 : \mathbb{N} \to \mathbb{N}$ such that

(A$_1$, A$_2$) $\forall x \in X,\ \bigcap_{n\in\mathbb{N}} V_n(x) = \{x\}$,
(A$_3$) $\forall n, m \in \mathbb{N},\ \forall x \in X,\ V_{\alpha_1(n,m)}(x) \subseteq V_n(x) \cap V_m(x)$,
(A$_4$) $\forall n \in \mathbb{N},\ \forall x, y \in X,\ x \in V_{\alpha_2(n)}(y) \Rightarrow y \in V_n(x)$,
(A$_5$) $\forall n \in \mathbb{N},\ \forall x, y \in X,\ x \in V_{\alpha_3(n)}(y) \Rightarrow V_{\alpha_3(n)}(x) \subseteq V_n(y)$.

$\mathbf{U} := (X, \{V_n\}, \alpha_1, \alpha_2, \alpha_3)$ is called an *effective uniform topological space*. The topology of $X$ generated by the uniformity $\{V_n\}$ is denoted by $\tau_{\mathbf{U}}$.

**Remark 4.1.** $(X, \tau_{\mathbf{U}})$ is a topological space with $\{V_n(x)\}$ as a neighborhood system. So for a subset $A \subseteq X$, the interior of $A$ in $(X, \tau_{\mathbf{U}})$, denoted by $A^\circ$, is defined as follows:

$$\forall x \in X \ \left(x \in A^\circ :\Leftrightarrow \exists n \in \mathbb{N},\ V_n(x) \subseteq A\right).$$

In particular, if $x \in V_{\alpha_2(\alpha_3(n))}(y)$, then by (A$_4$) and (A$_5$), $V_{\alpha_3(n)}(y) \subseteq V_n(x)$, and so $y \in V_n(x)^\circ$.

**Example 4.1** (*Metric spaces*). Let $(M, d)$ be a metric space. For $n \in \mathbb{N}$, define $V_n : X \to \mathcal{P}(X)$ by $V_n(x) := \left\{y \in M; d(x,y) < 2^{-n}\right\}$, and $\alpha_1 : \mathbb{N}^2 \to \mathbb{N}$ and $\alpha_2, \alpha_3 : \mathbb{N} \to \mathbb{N}$ by

$$\alpha_1(n,m) := \max\{n, m\}, \quad \alpha_2(n) := n, \quad \alpha_3(n) := n+1.$$

Then $\mathbf{M} := (M, \{V_n\}, \alpha_1, \alpha_2, \alpha_3)$ is an effective uniform topological space.

In the rest of this paper, we assume that $\mathbf{U} = (X, \{V_n\}_{n\in\mathbb{N}}, \alpha_1, \alpha_2, \alpha_3)$ is an effective uniform topological space. We also assume the following assumption, i.e., the separability of $(X, \tau_{\mathbf{U}})$. This is needed to define a structure of an effective topological space.

**Assumption 4.1.** There is a sequence $\{e_i\}_{i\in\mathbb{N}} \subseteq X$ which is dense in $(X, \tau_{\mathbf{U}})$.

**Definition 4.2.** Define a triple $\mathbf{S} = (X, \sigma, \nu)$ as follows:

(1) $\sigma$ is a countable system of subsets of $(X, \tau_{\mathbf{U}})$ defined by

$$\sigma := \left\{V_n(e_i)^\circ;\, n, i \in \mathbb{N}\right\}.$$

(2) $\nu : \Sigma^* \rightharpoonup \sigma$ is a notation of $\sigma$ defined by

$$\begin{cases} \mathrm{dom}(\nu) := \mathrm{dom}(\nu^2_{\mathbb{N}}), \\ \nu(w) := V_n(e_i)^\circ \text{ for } w \in \mathrm{dom}(\nu) \text{ with } \nu^2_{\mathbb{N}}(w) = (n, i). \end{cases}$$

In many cases, effectivities of inclusion and membership relations are needed. Axiom $(A_1)$–$(A_5)$ are not enough to provide these effectivities. So we assume the following assumption: (1) states that $R(c, n, i, m, j)$ guarantees the inclusion $V_n(e_i) \subseteq V_m(e_j)$ with a 'certificate' $c$; (2) states that a 'certificate' $c$ and $n$ satisfying $R(c, n, i, m, j)$ can be found locally uniformly w.r.t. $e_i$.

**Assumption 4.2.** There is a 5-ary recursive relation $R$ such that

(1) $\forall c, n, i, m, j \in \mathbb{N},\ R(c, n, i, m, j) \Rightarrow V_n(e_i) \subseteq V_m(e_j)$,
(2) $\forall m, j \in \mathbb{N},\ \forall x \in V_m(e_j)^\circ,\ \exists N, c \in \mathbb{N}$,

$$\begin{cases} V_N(x) \subseteq V_m(e_j), \text{ and} \\ \forall\, i \in \mathbb{N}\ \left(e_i \in V_{\alpha_3(N)}(x) \Rightarrow R(c, \alpha_3(N), i, m, j)\right). \end{cases}$$

**Example 4.2** (*Computable metric spaces*). A triple $(M, d, \{e_i\}_{i\in\mathbb{N}})$ is called a *computable metric space*, if

(1) $d$ is a metric on $M$,
(2) $\{e_i\} \subseteq M$ is dense in $(M, d)$,
(3) $\left\{d(e_i, e_j)\right\}_{i,j\in\mathbb{N}}$ is a computable sequence of reals.

Let $(M, d, \{e_i\}_{i\in\mathbb{N}})$ be a computable metric space. Define an effective uniform topological space $\mathbf{M} = (M, \{V_n\}, \alpha_1, \alpha_2, \alpha_3)$ as in Example 4.1. $\mathbf{M}$ satisfies Assumption 4.1 trivially. We show that $\mathbf{M}$ satisfies (1)–(2) of Assumption 4.2. By Remark 2.1, there is a recursive sequence $\left\{r_{i,j,k}\right\}_{i,j,k\in\mathbb{N}} \subseteq \mathbb{Q}$ such that

$$\forall i, j, k \in \mathbb{N}, \quad \left|d(e_i, e_j) - r_{i,j,k}\right| \leqslant 2^{-k}.$$

Define a recursive relation $R$ by

$$R(c, n, i, m, j) :\Leftrightarrow 2^{-m} - 2^{-n} - r_{i,j,c} - 2^{-c} \geqslant 0.$$

(1) If $R(c, n, i, m, j)$ holds, then $V_n(e_i) \subseteq V_m(e_j)$, because for each $y \in V_n(e_i)$,

$$d(y, e_j) \leqslant d(y, e_i) + d(e_i, e_j) < 2^{-n} + r_{i,j,c} + 2^{-c} \leqslant 2^{-m}$$

and thus $y \in V_m(e_j)$. Therefore, (1) of Assumption 4.2 holds.

(2) Let $x \in V_m(e_j)^\circ$. Choose $N, c \in \mathbb{N}$ large enough so that

$$2^{-m} - d(x, e_j) \geqslant 2^{-N} + 2^{-c+1}.$$

Then $V_N(x) \subseteq V_m(e_j)$, because for each $y \in V_N(x)$,

$$d(y, e_j) \leqslant d(y, x) + d(x, e_j) < 2^{-N} + d(x, e_j) \leqslant 2^{-m},$$

and thus $y \in V_m(e_j)$. Note that $\alpha_3(N) = N + 1$. For $i \in \mathbb{N}$ with $e_i \in V_{N+1}(x)$,

$$\begin{aligned} 2^{-m} - 2^{-N-1} - r_{i,j,c} - 2^{-c} &\geqslant 2^{-m} - 2^{-N-1} - d(e_i, e_j) - 2^{-c+1} \\ &\geqslant 2^{-m} - 2^{-N-1} - d(e_i, x) - d(x, e_j) - 2^{-c+1} \\ &> 2^{-m} - 2^{-N} - d(x, e_j) - 2^{-c+1} \\ &\geqslant 0. \end{aligned}$$

So $R(c, N + 1, i, m, j)$ holds. Therefore, (2) of Assumption 4.2 holds.

The following lemma is a typical example of the application of Assumption 4.2. It follows from the lemma that $e_i \in V_m(e_j)^\circ$ is an r.e. relation of $i, m, j \in \mathbb{N}$.

**Lemma 4.1.** *Under Assumption* 4.2, *the following holds for each* $i, m, j \in \mathbb{N}$:

$$e_i \in V_m(e_j)^\circ \quad \Leftrightarrow \quad \exists c,\ n \in \mathbb{N},\ R(c, n, i, m, j).$$

**Proof.** Suppose that $e_i \in V_m(e_j)^\circ$. Apply (2) of Assumption 4.2 with $x = e_i$, and let $n := \alpha_3(N)$. Then $R(c, n, i, m, j)$ holds. Conversely, $R(c, n, i, m, j)$ guarantees $V_n(e_i) \subseteq V_m(e_j)$, and hence $e_i \in V_m(e_j)^\circ$. □

**Lemma 4.2.** *Under Assumption* 4.1, $\sigma$ *is a countable base of* $(X, \tau_{\mathbf{U}})$.

**Proof.** By the definition, $\sigma$ is a countable system of open sets of $(X, \tau_{\mathbf{U}})$. We show that it is a base of $(X, \tau_{\mathbf{U}})$, i.e., for each open set $O$ of $(X, \tau_{\mathbf{U}})$ and for each $x \in O$, there is $A \in \sigma$ such that $x \in A \subseteq O$. By Remark 4.1, there is $N \in \mathbb{N}$ such that $x \in V_N(x) \subseteq O$. Let $n := \alpha_3(N)$. By the density of $\{e_i\}$, there is $i \in \mathbb{N}$ such that $e_i \in V_{\alpha_2(\alpha_3(n))}(x) \cap V_n(x)$. Then, $x \in V_n(e_i)^\circ$ and $V_n(e_i) \subseteq V_N(x)$ by ($A_4$) and ($A_5$). So $x \in V_n(e_i)^\circ \subseteq O$. □

**Proposition 4.1.** *Under Assumption* 4.1 *and* 4.2,

(1) $\mathbf{S}$ *is an effective topological space and* $\tau_{\mathbf{S}} = \tau_{\mathbf{U}}$,
(2) $\{e_i\}$ *is* $\delta'_{\mathbf{S}}$*-computable, so* $\mathbf{S}$ *satisfies Assumption* 3.1.

**Remark 4.2.** By (1) of Proposition 4.1, Remark 3.1 and the definition of $\nu$, a sequence $\{x_i\}_{i \in \mathbb{N}^I} \subseteq X$ is $\delta'_{\mathbf{S}}$-computable if and only if $x_i \in V_m(e_j)^\circ$ is an r.e. relation of $i \in \mathbb{N}^I$ and $m, j \in \mathbb{N}$.

**Proof of Proposition 4.1.** (1) By Lemma 4.2, $\sigma$ is a countable base of $(X, \tau_{\mathbf{U}})$, which is a $T_0$-space by ($A_1$, $A_2$). Hence, $\mathbf{S} = (X, \sigma, \nu)$ is an effective topological space and $\tau_{\mathbf{S}} = \tau_{\mathbf{U}}$.
(2) By Lemma 4.1, for each $i, m, j \in \mathbb{N}$,

$$e_i \in V_m(e_j)^\circ \quad \Leftrightarrow \quad \exists c, \quad n \in \mathbb{N}, \quad R(c, n, i, m, j).$$

Since $R$ is recursive, the right-hand side is an r.e. relation of $i, m, j \in \mathbb{N}$. As in Remark 4.2, this means that $\{e_i\}$ is $\delta'_{\mathbf{S}}$-computable. Then, $\mathbf{S}$ satisfies Assumption 3.1, because $\mathrm{dom}(\nu) = \mathrm{dom}\left(\nu^2_{\mathbb{N}}\right)$ is r.e., and $\{e_i\}$ is dense in $(X, \tau_{\mathbf{S}})$ by Assumption 4.1 and (1) of Proposition 4.1. □

## 5. Computable sequences

In this section, we define computability of sequences of an effective uniform topological space $X$ (Definition 5.3) by using effective convergence. It is proved that the set of the computable sequences forms a computability structure (Proposition 5.1), and that $\delta'_{\mathbf{S}}$-computable sequences provide the same set of sequences (Theorem 5.1).

First, we recall the definitions of effective convergence and computability structures on effective uniform topological spaces [9,12].

**Definition 5.1** (*Effective convergence, Yasugi et al. [12]*). Let $\{x_{i,k}\}_{i\in\mathbb{N}^I,\, k\in\mathbb{N}}$ and $\{x_i\}_{i\in\mathbb{N}^I}$ be sequences of $X$. We say $\{x_{i,k}\}$ *converges to* $\{x_i\}$ *V-effectively as* $k \to \infty$, if there is a recursive function $\mu : \mathbb{N}^I \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall i \in \mathbb{N}^I,\ \ \forall n \in \mathbb{N},\ \ \forall k \geqslant \mu(i,n), \quad x_{i,k} \in V_n(x_i).$$

**Definition 5.2** (*Computability structure, Yasugi et al. [12]*). A set $\mathcal{C}$ of sequences of $X$ is called a *computability structure*, if the followings holds:

($C_1$) $\mathcal{C}$ is non-empty,
($C_2$) If $\{x_i\}_{i\in\mathbb{N}^I} \in \mathcal{C}$ and $\lambda : \mathbb{N}^J \to \mathbb{N}^I$ is recursive, then $\{x_{\lambda(j)}\}_{j\in\mathbb{N}^J} \in \mathcal{C}$,
($C_3$) If $\{x_{i,k}\}_{i\in\mathbb{N}^I,\, k\in\mathbb{N}} \in \mathcal{C}$ and it converges to $\{x_i\}_{i\in\mathbb{N}^I} \subseteq X$ $V$-effectively as $k \to \infty$, then $\{x_i\} \in \mathcal{C}$.

Now, we define computability of sequences of $X$. The computable sequences are defined along with the separating set $\{e_i\}$ by using effective convergence. This is a generalization of computable sequence of reals in [7].

**Definition 5.3** (*Computable sequence*).

(1) A sequence $\{r_i\}_{i\in\mathbb{N}^I} \subseteq X$ is called *recursive w.r.t. e*, if there is a recursive function $\lambda : \mathbb{N}^I \to \mathbb{N}$ such that $r_i = e_{\lambda(i)}$ for each $i \in \mathbb{N}^I$.
(2) A sequence $\{x_i\}_{i\in\mathbb{N}^I} \subseteq X$ is called *V-computable w.r.t. e*, if there is a recursive sequence $\{r_{i,k}\}_{i\in\mathbb{N}^I,\, k\in\mathbb{N}} \subseteq X$ w.r.t. $e$ which converges to $\{x_i\}$ $V$-effectively as $k \to \infty$.

**Proposition 5.1.** *Under Assumption* 4.1, *the set* $\mathcal{C}$ *of V-computable sequences w.r.t. e is a computability structure.*

**Proof.** All recursive sequences w.r.t. $e$ are clearly $V$-computable w.r.t. $e$, and thus ($C_1$) holds. ($C_2$) is easy. So we prove ($C_3$).

Suppose that $\{x_{i,k}\}_{i\in\mathbb{N}^I,\, k\in\mathbb{N}} \in \mathcal{C}$ and it converges to $\{x_i\}_{i\in\mathbb{N}^I} \subseteq X$ $V$-effectively as $k \to \infty$. Let $\mu : \mathbb{N}^I \times \mathbb{N} \to \mathbb{N}$ be a recursive function as in Definition 5.1. By $\{x_{i,k}\} \in \mathcal{C}$, there is a recursive sequence $\{r_{i,k,l}\}_{i\in\mathbb{N}^I,\, k,l\in\mathbb{N}}$ w.r.t. $e$ which converges to $\{x_{i,k}\}$ $V$-effectively as $l \to \infty$. Then there is a recursive function $\mu' : \mathbb{N}^I \times \mathbb{N}^2 \to \mathbb{N}$ such that

$$\forall i \in \mathbb{N}^I,\ \ \forall k,n \in \mathbb{N},\ \ \forall l \geqslant \mu'(i,k,n), \quad r_{i,k,l} \in V_n(x_{i,k}). \tag{1}$$

We can assume that $\mu'(i,k,n) \geqslant \mu'(i,k,m)$ if $n \geqslant m$. Define a recursive sequence $\{s_{i,k}\}_{i\in\mathbb{N}^I,\, k\in\mathbb{N}}$ w.r.t. $e$ and a recursive function $\mu'' : \mathbb{N}^I \times \mathbb{N} \to \mathbb{N}$ by

$$s_{i,k} := r_{i,\,k,\,\mu'(i,k,k)}, \quad \mu''(i,n) := \max\{\alpha_3(n),\ \mu(i, \alpha_3(n))\}.$$

Let $i \in \mathbb{N}^I$, $n \in \mathbb{N}$ and $k \geqslant \mu''(i,n)$. Then, $k \geqslant \alpha_3(n)$, and thus $\mu'(i,k,k) \geqslant \mu'(i,k,\alpha_3(n))$. Hence, by (1), $s_{i,k} \in V_{\alpha_3(n)}(x_{i,k})$ holds. By $k \geqslant \mu(i,\alpha_3(n))$, $x_{i,k} \in V_{\alpha_3(n)}(x_i)$, and by ($A_5$), $V_{\alpha_3(n)}(x_{i,k}) \subseteq V_n(x_i)$. So $s_{i,k} \in V_n(x_i)$ for each $k \geqslant \mu''(i,n)$. This means that $\{x_i\}$ is $V$-computable w.r.t. $e$. □

**Theorem 5.1.** *Under Assumptions* 4.1 *and* 4.2, *for each sequence* $\{x_i\}_{i\in\mathbb{N}^I}$ *of* $X$, $\{x_i\}$ *is* $\delta'_{\mathbf{S}}$*-computable if and only if* $\{x_i\}$ *is* $V$*-computable w.r.t.* $e$.

**Proof.** (If part): Suppose that $\{x_i\}$ is $V$-computable w.r.t. $e$. Then, there are recursive functions $\lambda, \mu : \mathbb{N}^I \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall i \in \mathbb{N}^I,\ \ \forall n \in \mathbb{N},\ \ \forall k \geqslant \mu(i,n), \quad e_{\lambda(i,k)} \in V_n(x_i). \tag{2}$$

**Claim 1.** *Define a recursive function* $\kappa : \mathbb{N}^2 \to \mathbb{N}$ *by*

$$\kappa(i,n) := \mu(i, \alpha_1(n, \alpha_2(\alpha_3(n)))).$$

*Then for each* $i \in \mathbb{N}^I$ *and* $m, j \in \mathbb{N}$,

$$x_i \in V_m(e_j)^\circ \ \Leftrightarrow\ \exists c, n \in \mathbb{N},\ R(c, n, \lambda(i, \kappa(i,n)), m, j). \tag{3}$$

**Proof.** ($\Rightarrow$): Suppose that $x_i \in V_m(e_j)^\circ$. Then by (2) of Assumption 4.2, there are $N, c \in \mathbb{N}$ such that

$$\forall h \in \mathbb{N}, \quad e_h \in V_{\alpha_3(N)}(x_i) \Rightarrow R(c, \alpha_3(N), h, m, j).$$

Let $n := \alpha_3(N)$ and $h := \lambda(i, \kappa(i,n))$. Then by (2) and ($A_3$),

$$e_h \in V_{\alpha_1(n,\alpha_2(\alpha_3(n)))}(x_i) \subseteq V_n(x_i).$$

So by ($A_5$), $V_n(e_h) \subseteq V_N(x_i)$. Therefore $R(c,n,h,m,j)$ holds.

($\Leftarrow$): Suppose that the right-hand side of (3) holds. Let $h := \lambda(i, \kappa(i,n))$. Then by (2) and ($A_3$),

$$e_h \in V_{\alpha_1(n,\alpha_2(\alpha_3(n)))}(x_i) \subseteq V_{\alpha_2(\alpha_3(n))}(x_i).$$

So by ($A_4$) and ($A_5$), $x_i \in V_n(e_h)^\circ$. Furthermore, $R(c,n,h,m,j)$ guarantees $V_n(e_h) \subseteq V_m(e_j)$. Therefore, $x_i \in V_m(e_j)^\circ$ holds. □

By recursiveness of $R$, the right-hand side of (3) is an r.e. relation of $i \in \mathbb{N}^I$ and $m, j \in \mathbb{N}$, and this proves that $\{x_i\}$ is $\delta'_{\mathbf{S}}$-computable.

(Only if part): Suppose that $\{x_i\}_{i\in\mathbb{N}^I}$ is $\delta'_{\mathbf{S}}$-computable. By Remark 4.2,

$$\forall m \leqslant k, \quad x_i \in V_{\alpha_2(m)}(e_j)^\circ \tag{4}$$

is an r.e. relation of $i \in \mathbb{N}^I$ and $k, j \in \mathbb{N}$. By (A$_3$)–(A$_5$), and the density of $\{e_i\}$, for each $i \in \mathbb{N}^I$ and $k \in \mathbb{N}$, there is $j \in \mathbb{N}$ which satisfies (4). So there is a recursive function $\lambda : \mathbb{N}^I \times \mathbb{N} \to \mathbb{N}$ such that for each $i \in \mathbb{N}^I$ and $k \in \mathbb{N}$,

$$\forall m \leqslant k, \quad x_i \in V_{\alpha_2(m)}(e_{\lambda(i,k)})^\circ.$$

Then, for each $i \in \mathbb{N}^I$, $n \in \mathbb{N}$ and $k \geqslant n$, $x_i \in V_{\alpha_2(n)}(e_{\lambda(i,k)})$ holds, and so $e_{\lambda(i,k)} \in V_n(x_i)$ by (A$_4$). Hence $\{e_{\lambda(i,k)}\}_{i \in \mathbb{N}^I,\, k \in \mathbb{N}}$ converges to $\{x_i\}$ $V$-effectively as $k \to \infty$. Therefore, $\{x_i\}$ is $V$-computable w.r.t. $e$. □

It is natural to define sequential computability by $V$-computable sequences w.r.t. $e$ and computable sequences of reals. So we propose the following definition.

**Definition 5.4.** A function $f : X \to \mathbb{R}$ is called *sequentially V-computable w.r.t. e*, if $f$ maps each $V$-computable sequence w.r.t. $e$ to a computable sequence of reals.

By Theorem 5.1, the following holds.

**Corollary 5.1.** *Under Assumptions* 4.1 *and* 4.2, *for each function* $f : X \to \mathbb{R}$, *$f$ is sequentially* $\delta'_{\mathbf{S}}$*-computable if and only if $f$ is sequentially V-computable w.r.t. e.*

## 6. Effective continuities

In this section, we define three different notions of effective continuity on effective uniform topological spaces (Definitions 6.1, 6.2, 6.3). We assume the computability of finite union of neighborhoods (Assumption 6.1) to compare these effective continuities. The effective continuities and their relations discussed in this section are as follows (Propositions 6.1, 6.2 and Theorem 6.1):

$V$-effectively uniformly continuous ($V$-EUC)
$\Rightarrow$ $V$-effectively locally uniformly continuous ($V$-ELUC) w.r.t. $e$
$\Rightarrow$ $V$-effectively continuous ($V$-EC) w.r.t. $e$
$\Leftrightarrow$ Effectively continuous (EC) w.r.t. $\beta$ (defined in Section 3).

**Definition 6.1** (*V-Effective uniform continuity*). A function $f : X \to \mathbb{R}$ is called *V-effectively uniformly continuous* (*V-EUC*), if there is a recursive function $\mu : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N}, \ \ \forall x, y \in X, \ \ y \in V_{\mu(k)}(x) \Rightarrow |f(x) - f(y)| \leqslant 2^{-k}.$$

**Definition 6.2** (*V-Effective local uniform continuity w.r.t. e*). A function $f : X \to \mathbb{R}$ is called *V-effectively locally uniformly continuous (V-ELUC) w.r.t. e*, if there are recursive functions $\mu : \mathbb{N}^2 \to \mathbb{N}$ and $\gamma : \mathbb{N} \to \mathbb{N}$ such that

(1) $\bigcup_{j \in \mathbb{N}} V_{\gamma(j)}(e_j)^\circ = X$,
(2) $\forall j, k \in \mathbb{N}, \ \forall x, y \in V_{\gamma(j)}(e_j), \ y \in V_{\mu(j,k)}(x) \Rightarrow |f(x) - f(y)| \leqslant 2^{-k}$.

**Definition 6.3** (*V-Effective continuity w.r.t. e*). A function $f : X \to \mathbb{R}$ is called *V-effectively continuous* (*V-EC*) *w.r.t. e*, if there is a recursive function $\gamma : \mathbb{N}^2 \to \mathbb{N}$ such that

(1) $\forall k \in \mathbb{N}, \ \bigcup_{j \in \mathbb{N}} V_{\gamma(j,k)}(e_j)^\circ = X$,

(2) $\forall j, k \in \mathbb{N},\ \forall x, y \in V_{\gamma(j,k)}(e_j),\ |f(x) - f(y)| \leqslant 2^{-k}$.

**Proposition 6.1.** *Under Assumption* 4.1, *for each function* $f : X \to \mathbb{R}$, *if f is V-EUC*, *then f is V-ELUC w.r.t. e*.

**Proof.** Suppose that $f$ is $V$-EUC. Let $\mu : \mathbb{N} \to \mathbb{N}$ be a recursive function as in Definition 6.1. Define recursive functions $\mu' : \mathbb{N}^2 \to \mathbb{N}$ and $\gamma' : \mathbb{N} \to \mathbb{N}$ by

$$\mu'(i, k) := \mu(i), \quad \gamma'(i) := 0.$$

By ($A_4$) and the density of $\{e_i\}$, for each $x \in X$, $x \in V_0(e_i)$ holds for some $i \in \mathbb{N}$. So $\bigcup_{i\in\mathbb{N}} V_{\gamma'(i)}(e_i) = X$. Property (2) of Definition 6.2 follows from the property of $\mu$ and the definition of $\mu'$. Therefore $f$ is $V$-ELUC w.r.t. $e$. □

To prove the following proposition and theorem, we need to construct a new open covering from the old one provided by the properties of effective continuities. The biggest difficulty is to make it a covering of $X$. To do this, we have to take open sets as large as possible. So we need to assume the computability of finite union of the neighborhoods.

**Assumption 6.1.** There is a recursive function $\alpha_4 : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\forall n, m \in \mathbb{N},\ \forall x \in X, \quad V_{\alpha_4(n,m)}(x) = V_n(x) \cup V_m(x).$$

**Example 6.1** (*Metric spaces*). Assumption 6.1 holds for **M** in Example 4.1, because $\alpha_4(n, m) := \min\{n, m\}$ satisfies the condition of the assumption.

**Remark 6.1.** $\alpha_1$ and $\alpha_4$ will be extended to recursive functions on $\mathcal{F}(\mathbb{N})$ so that the following properties hold for each $F \in \mathcal{F}(\mathbb{N})$ and $x \in X$:

$$V_{\alpha_1(F)}(x) \subseteq \bigcap_{n\in F} V_n(x), \quad V_{\alpha_4(F)}(x) = \bigcup_{n\in F} V_n(x).$$

**Proposition 6.2.** *Under Assumptions* 4.1, 4.2 *and* 6.1, *for each function* $f : X \to \mathbb{R}$, *if f is V-ELUC w.r.t. e*, *then f is V-EC w.r.t. e*.

The idea of the proof is as follows. Suppose that $f$ is $V$-ELUC w.r.t. $e$, and let $\mu : \mathbb{N}^2 \to \mathbb{N}$ and $\gamma : \mathbb{N} \to \mathbb{N}$ be recursive functions as in Definition 6.2. Assume that $\gamma' : \mathbb{N}^2 \to \mathbb{N}$ satisfies the following equation:

$$V_{\gamma'(i,k)}(e_i) = \bigcup \left\{ V_n(e_i) \cap V_{\mu(j,k+1)}(e_i); n, j \in \mathbb{N} \text{ such that } V_n(e_i) \subseteq V_{\gamma(j)}(e_j) \right\}.$$

Then $\gamma'$ satisfies properties (1)–(2) of Definition 6.3. But this equation may not hold for any recursive $\gamma'$. So we need to modify it so that it is satisfied by some recursive function.

**Proof of Proposition 6.2.** Suppose that $f$ is $V$-ELUC w.r.t. $e$, and let $\mu : \mathbb{N}^2 \to \mathbb{N}$ and $\gamma : \mathbb{N} \to \mathbb{N}$ be recursive functions as in Definition 6.2.

**Claim 1.** $\forall i \in \mathbb{N},\ \exists c, n, j \in \mathbb{N},\ R(c, n, i, \gamma(j), j)$.

**Proof.** By property (1) of Definition 6.2, for each $i \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $e_i \in V_{\gamma(j)}(e_j)^\circ$. Then by (2) of Assumption 4.2, there are $c, n \in \mathbb{N}$ such that $R(c, n, i, \gamma(j), j)$ holds. $\square$

Since $R$ is recursive, there is a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that

$$\forall i \in \mathbb{N}, \ \exists c, n, j \leqslant g(i), \quad R(c, n, i, \gamma(j), j). \tag{5}$$

We can assume that $g(i) \geqslant i$ for each $i \in \mathbb{N}$. Define a recursive function $\gamma' : \mathbb{N}^2 \to \mathbb{N}$ by

$$\gamma'(i, k) := \alpha_4 \ \{\alpha_1(n, \mu(j, k+1)); c, n, j \leqslant g(i) \text{ such that } R(c, n, i, \gamma(j), j)\}.$$

This is well-defined, because the argument of $\alpha_4$ is non-empty by (5).

**Claim 2.** $\forall k \in \mathbb{N}, \ \bigcup_{i \in \mathbb{N}} V_{\gamma'(i,k)}(e_i)^\circ = X.$

**Proof.** Let $k \in \mathbb{N}$ and $x \in X$. We show that $x \in V_{\gamma'(i,k)}(e_i)^\circ$ for some $i \in \mathbb{N}$. If $x$ is an isolated point of $(X, \tau_{\mathbf{U}})$, this is trivial, because $x = e_i$ for some $i \in \mathbb{N}$. So we assume that $x$ is an accumulation point. By property (1) of Definition 6.2, there is $j \in \mathbb{N}$ such that $x \in V_{\gamma(j)}(e_j)^\circ$. Then by (2) of Assumption 4.2, there are $N, c \in \mathbb{N}$ such that

$$\forall i \in \mathbb{N}, \quad e_i \in V_{\alpha_3(N)}(x) \Rightarrow R(c, \alpha_3(N), i, \gamma(j), j). \tag{6}$$

Let $n := \alpha_3(N)$ and $m := \alpha_1(n, \mu(j, k+1))$. By the density of $\{e_i\}$, there is $i \in \mathbb{N}$ such that $e_i \in V_n(x) \cap V_{\alpha_2(\alpha_3(m))}(x)$. We can assume that $i$ is large enough so that $c, n, j \leqslant g(i)$, because $x$ is an accumulation point. Since $e_i \in V_n(x)$, $R(c, n, i, \gamma(j), j)$ holds by (6). So $V_m(e_i) \subseteq V_{\gamma'(i,k)}(e_i)$ by the definition of $\gamma'$ and the property of $\alpha_4$. Since $e_i \in V_{\alpha_2(\alpha_3(m))}(x)$, $x \in V_m(e_i)^\circ$ by ($A_4$) and ($A_5$). Therefore $x \in V_m(e_i)^\circ \subseteq V_{\gamma'(i,k)}(e_i)^\circ$. $\square$

**Claim 3.** $\forall i, k \in \mathbb{N}, \ \forall x, y \in V_{\gamma'(i,k)}(e_i), \ |f(x) - f(y)| \leqslant 2^{-k}.$

**Proof.** Let $i, k \in \mathbb{N}$ and $x \in V_{\gamma'(i,k)}(e_i)$. Then by the definition of $\gamma'$ and the property of $\alpha_4$, there are $c, n, j \leqslant g(i)$ such that $R(c, n, i, \gamma(j), j)$ holds and $x_i \in V_{\alpha_1(n,\mu(j,k+1))}(e_i)$. Then $x \in V_n(e_i) \cap V_{\mu(j,k+1)}(e_i)$ by ($A_3$), and $V_n(e_i) \subseteq V_{\gamma(j)}(e_j)$ by $R(c, n, i, \gamma(j), j)$. So $x, e_i \in V_{\gamma(j)}(e_j)$ and $x \in V_{\mu(j,k+1)}(e_i)$. Hence by property (2) of Definition 6.2, $|f(x) - f(e_i)| \leqslant 2^{-k-1}$. Therefore, for each $x \in V_{\gamma'(i,k)}(e_i)$, $|f(x) - f(e_i)| \leqslant 2^{-k-1}$. This proves Claim 3. $\square$

From Claims 2 and 3, it follows that $\gamma'$ satisfies the properties of Definition 6.3. Therefore $f$ is $V$-EC w.r.t. $e$. $\square$

The following theorem can be proved by the same argument as in the proof of Proposition 6.2.

**Theorem 6.1.** *Define a countable base $\beta$ of $(X, \tau_{\mathbf{U}})$ and its notation $\nu_\beta$ as in Definition* 3.3. *Under Assumptions* 4.1, 4.2 *and* 6.1, *for each function $f : X \to \mathbb{R}$, $f$ is EC w.r.t. $\beta$ if and only if $f$ is V-EC w.r.t. $e$.*

**Proof.** The if part is trivial. So we prove the only if part. Suppose that $f$ is EC w.r.t. $\beta$, and let $\gamma : \mathbb{N}^2 \to \Sigma^*$ be a recursive function as in Definition 3.4. We can assume that $\{w; \imath(w) \lhd \gamma(j, k)\} \neq \emptyset$ for each $j, k \in \mathbb{N}$. For $w \in \mathrm{dom}(\nu)$ with $\nu^2_{\mathbb{N}}(w) = (m, j)$, we write $R(c, n, i, m, j)$ as $R(c, n, i, w)$ for short.

**Claim 1.** $\forall i, k \in \mathbb{N},\ \exists j \in \mathbb{N},\ \forall \imath(w) \lhd \gamma(j,k),\ \exists c_w, n_w \in \mathbb{N},\ R(c_w, n_w, i, w)$.

**Proof.** By property (2) of Definition 3.4, for each $i, k \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $e_i \in \nu_\beta(\gamma(j,k))$. Then for each $\imath(w) \lhd \gamma(j,k)$, $e_i \in \nu(w)$. So by (2) of Assumption 4.2, there are $c_w, n_w \in \mathbb{N}$ such that $R(c_w, n_w, i, w)$ holds. $\square$

Since $R$ is recursive, there is a recursive function $g : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\forall i, k \in \mathbb{N},\ \ \exists j \leqslant g(i,k),\ \ \forall \imath(w) \lhd \gamma(j,k),\ \ \exists c_w, n_w \leqslant g(i,k),$$
$$R(c_w, n_w, i, w). \tag{7}$$

We can assume that $g(i,k) \geqslant i$ for each $i, k \in \mathbb{N}$. Define a recursive function $\gamma' : \mathbb{N}^2 \to \mathbb{N}$ by

$$\begin{aligned} \gamma'(i,k) := \alpha_4 \{ & \alpha_1 \{\alpha_3(n_w); \imath(w) \lhd \gamma(j,k)\}; \\ & j, c_w, n_w \leqslant g(i,k) (\imath(w) \lhd \gamma(j,k)) \text{ such that} \\ & \forall \imath(w) \lhd \gamma(j,k),\ R(c_w, n_w, i, w) \}. \end{aligned}$$

This is well-defined, because the arguments of $\alpha_1$ and $\alpha_4$ are non-empty by the assumption of $\gamma$ and (7).

**Claim 2.** $\forall k \in \mathbb{N},\ \bigcup_{i \in \mathbb{N}} V_{\gamma'(i,k)}(e_i)^\circ = X$.

**Proof.** Let $k \in \mathbb{N}$ and $x \in X$. We show that $x \in V_{\gamma'(i,k)}(e_i)^\circ$ for some $i \in \mathbb{N}$. If $x$ is an isolated point of $(X, \tau_{\mathbf{U}})$, this is trivial, because $x = e_i$ for some $i \in \mathbb{N}$. So we assume that $x$ is an accumulation point. By property (2) of Definition 3.4, there is $j \in \mathbb{N}$ such that $x \in \nu_\beta(\gamma(j,k))$. Then for each $\imath(w) \lhd \gamma(j,k)$, $x \in \nu(w)$. So by (2) of Assumption 4.2, there are $N_w, c_w \in \mathbb{N}$ such that

$$\forall i \in \mathbb{N}, \quad e_i \in V_{\alpha_3(N_w)}(x) \Rightarrow R(c_w, \alpha_3(N_w), i, w). \tag{8}$$

Let $n_w := \alpha_3(N_w)$ for each $\imath(w) \lhd \gamma(j,k)$ and

$$m := \alpha_1 \{\alpha_3(n_w); \imath(w) \lhd \gamma(j,k)\}.$$

By the density of $\{e_i\}$, there is $i \in \mathbb{N}$ such that

$$e_i \in \bigcap_{\imath(w) \lhd \gamma(j,k)} V_{n_w}(x) \cap V_{\alpha_2(\alpha_3(m))}(x).$$

We can assume that $i$ is large enough so that $j, c_w, n_w \leqslant g(i,k)$ for each $\imath(w) \lhd \gamma(j,k)$, because $x$ is an accumulation point. For each $\imath(w) \lhd \gamma(j,k)$, $e_i \in V_{n_w}(x)$. So $R(c_w, n_w, i, w)$ holds by (8). Hence, $V_m(e_i) \subseteq V_{\gamma'(i,k)}(e_i)$ by the definition of $\gamma'$ and the property of $\alpha_4$. Since $e_i \in V_{\alpha_2(\alpha_3(m))}(x)$, $x \in V_m(e_i)^\circ$ by ($A_4$) and ($A_5$). Therefore $x \in V_m(e_i)^\circ \subseteq V_{\gamma'(i,k)}(e_i)^\circ$. $\square$

**Claim 3.** $\forall i, k \in \mathbb{N},\ \forall x, y \in V_{\gamma'(i,k)}(e_i),\ |f(x) - f(y)| \leqslant 2^{-k+1}$.

**Proof.** Let $i, k \in \mathbb{N}$ and $x \in V_{\gamma'(i,k)}(e_i)$. Then by the definition of $\gamma'$ and the property of $\alpha_4$, there are $j \leqslant g(i,k)$ and $c_w, n_w \leqslant g(i,k)$ $(\imath(w) \lhd \gamma(j,k))$ such that for each $\imath(w) \lhd \gamma(j,k)$, $R(c_w, n_w, i, w)$ holds, and $x \in V_m(e_i)$, where

$$m := \alpha_1 \{\alpha_3(n_w); \imath(w) \lhd \gamma(j,k)\}.$$

Then for each $\imath(w) \lhd \gamma(j,k)$, $x \in V_{\alpha_3(n_w)}(e_i)$ by ($A_3$), and so $x \in V_{n_w}(e_i)^\circ$ by ($A_5$). Furthermore, $V_{n_w}(e_i)^\circ \subseteq v(w)$ by $R(c, n_w, i, w)$. So $x, e_i \in v(w)$ for each $\imath(w) \lhd \gamma(j,k)$, and thus $x, e_i \in v_\beta(\gamma(i,k))$. Hence by property (3) of Definition 3.4, $|f(x) - f(e_i)| \leqslant 2^{-k}$. Therefore, for each $x \in V_{\gamma'(i,k)}(e_i)$, $|f(x) - f(e_i)| \leqslant 2^{-k}$. This proves Claim 3. $\square$

Define $\gamma'' : \mathbb{N}^2 \to \mathbb{N}$ by $\gamma''(i,k) := \gamma'(i, k+1)$ for each $i, k \in \mathbb{N}$. From Claims 2 and 3, it follows that $\gamma''$ satisfies the properties of Definition 6.3. Therefore, *f* is *V*-EC w.r.t. *e*. $\square$

By Theorems 3.1, 6.1 and Corollary 5.1, $(\delta'_{\mathbf{S}}, \rho_{\mathrm{E}})$-computability can be characterized as sequential *V*-computability and *V*-EC w.r.t. *e* without using Type-2 computability.

**Corollary 6.1.** *Under Assumptions* 4.1, 4.2 *and* 6.1, *for each function* $f : X \to \mathbb{R}$, *f is* $(\delta'_{\mathbf{S}}, \rho_{\mathrm{E}})$-*computable if and only if f is sequentially V-computable and V-EC w.r.t. e.*

## Acknowledgments

The author thanks Takakazu Mori, Yoshiki Tsujii, Mariko Yasugi and two anonymous referees for valuable comments. This work has been supported in part by COE RA Grant of Tohoku University.

## References

[1] V. Brattka, Some notes on fine computability, J. UCS 8 (3) (2002) 382–395.
[3] A. Grzegorczyk, On the definitions of computable real continuous functions, Fund. Math. 44 (1957) 61–71.
[4] P. Hertling, Effectivity and effective continuity of functions between computable metric spaces, in: Combinatorics, Complexity, & Logic (Auckland, 1996), Springer Series in Discrete Mathematical Theoretical Computer Science, Springer, Singapore, 1997, pp. 264–275.
[5] T. Mori, On the computability of Walsh functions, Theoret. Comput. Sci. 284 (2) (2002) 419–436 Computability and Complexity in Analysis (Castle Dagstuhl, 1999).
[6] T. Mori, Y. Tsujii, M. Yasugi, Fine computable functions and effective fine convergence, in: T. Grubba, P. Hertling, H. Tsuiki, K. Weihrauch (Eds.), CCA 2005—Second International Conference on Computability and Complexity in Analysis, August 25–29, 2005, Kyoto, Japan, Informatik Berichte, vol. 326-7/2005, FernUniversität Hagen, Germany, 2005, pp. 177–197.
[7] M.B. Pour-El, J. Ian Richards, Computability in Analysis and Physics, Perspectives in Mathematical Logic, Springer, Berlin, 1989.
[8] D. Spreen, Representations versus numberings: on the relationship of two computability notions, Theoret. Comput. Sci. 262 (1–2) (2001) 473–499.
[9] Y. Tsujii, M. Yasugi, T. Mori, Some properties of the effective uniform topological space, in: J. Blanck, V. Brattka, P. Hertling (Eds.), CCA, Lecture Notes in Computer Science, vol. 2064, Springer, Berlin, 2000, pp. 336–356.
[10] Y. Tsujii, M. Yasugi, T. Mori, Sequential computability of a function: diagonal space and limiting recursion, Electron. Notes Theoret. Comput. Sci. 120 (2005) 187–199.
[11] K. Weihrauch, Computable analysis, Texts in Theoretical Computer Science, An EATCS Series, Springer, Berlin, 2000 (An introduction).
[12] M. Yasugi, T. Mori, Y. Tsujii, Effective sequence of uniformities and its effective limit, in: T. Grubba, P. Hertling, H. Tsuiki, K. Weihrauch (Eds.), CCA 2005—Second International Conference on Computability and Complexity in Analysis, August 25–29, 2005, Kyoto, Japan, Informatik Berichte, vol. 326-7/2005, FernUniversität Hagen, Germany, 2005, pp. 301–318.